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N-Slit Diffraction Pattern

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The diffraction pattern for particles passing through N slits is obtained by a truncation assumption on the Gaussian wave function when passing the slits and by pure wave propagation governed by the Schrödinger equation with boundaries. The standard limiting situations give the usual results. In case of a great number of equal and equally separated slits the calculations are performed by considering also the energy band structure induced by the periodic boundary geometry on the initial conditions. The results, which are qualitatively satisfactory in both cases, do not agree completely in general nor in the limiting cases.

1. INTRODUCTION

The study of the diffraction pattern for wave packets through slits is a central means to reveal the distinguishing properties of quantum mechanics (QM) (Feynman and Hibbs, 1965) such as the superposition principle of physical states. It represents also an interesting situation in which to compare the predictions of QM with those of other physical theories. In this connection explicit calculations have been done to compare the predictions of QM with those of the recent stochastic electrodynamics (SED) with spin (Cavalleri, 1997). The results, which have been obtained for one and two slits in the two-dimensional case (Zecca and Cavalleri, 1997; Zecca, 1999), make use, in a general mathematical framework, of an incoming Gaussian wave packet to describe a beam of particles. The wave packet is assumed to be truncated by the barriers when passing the slits and its time evolution after the slits is approximated by a factorized Schrödinger-like time evolution. By neglecting possible electromagnetic and spin interaction of the electron with the slits, a general diffraction pattern has been obtained that has the usual behavior

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for beam both narrow and large with respect to the slit aperture. By considering also what seems to be the dominant interaction, that is, the interaction of the electron with its image charge on the well of the slits, one finds, as already predicted by SED plus spin (Zecca and Cavalleri, 1997), the existence of lateral maxima on the QM diffraction pattern corresponding to the edges of the slits. The quantitative relevance of that effect, even considering the spin of the electron, is not decidable in the context of the theory and would require experimental verification.

The mentioned treatment easily generalizes to the N-slit case, which is done in the first part of this paper. In case of N large (practically infinite) and for equal and equally spaced slits, a problem arises that seems to be of interest. In the mentioned description the wave packet moves in the x, y plane parallel to the x axis and has a nontrivial probability distribution of the momentum in the y direction along which the N slits are disposed.

Accordingly, near the slits, the particle is subjected, in the y direction, to a nontrivial periodic potential, whatever kind of interaction one considers. When it emerges after the slits, the particle must therefore have an energy distribution in the y direction compatible with the energy band structure characteristic of a one-dimensional motion in a periodic potential.

It is the object of the second part of the paper to establish the effect of the periodic potential on the diffraction pattern. Following the general scheme (Zecca and Cavalleri, 1997), the particle is considered to be subjected to a periodic potential which is assumed to be the infinite limit of a finite periodic potential barrier. The explicit momentum dependence of the energy in the y direction is then determined from limiting values of well-known results concerning the energy band spectrum in the case of a one-dimensional particle. This information, together with the truncation assumption, is put into the initial condition of the wave packet, which is still assumed to evolve in a factorized form after the slits according to the free Schrödinger dynamics. The diffraction pattern that is obtained in general is different according as one considers periodic potentials or not. In the limiting cases of a beam of particles large or narrow with respect to the slit aperture, the difference of the profile of the diffraction pattern is more evident. Since the considerations relative to the periodic potential cannot *a priori* be ruled out, being the consequence of the geometry of the boundaries, an experimental verification of the results would be of interest.

2. *N*-SLIT DIFFRACTION FROM SCHRÖDINGER QM

The diffraction problem is considered in the two-dimensional case. The geometry of the slits is such that the region S which is inaccessible to the particle is the subset of the (x, y) plane defined by

$$S = \{(x, y) \| |x| < \delta, y \in \bigcup_{i=0}^{N} (d_i + b_i, d_{i+1})\} \tag{1}$$

where $d_0 = -\infty$, $b_i > 0$, $d_{N+1} = \infty$, $d_i + b_i < d_{i+1}$, $i = 1, 2, \ldots, N$. The slits therefore have an aperture b_i and the contiguous slits b_{i+1} and b_i are separated by a single barrier whose dimension in the y direction is $d_{i+1} - d_i - b_i$. The depth of the barriers is taken constant and equal to 2δ. For our consideration one can choose $\delta \to 0$. In the y direction, the set of the slits is given by the multi-interval set

$$I = \bigcup_{j=0}^{N} [d_j, d_j + b_j] \tag{2}$$

The motion of the particle in the above geometry can be described as that of a free Schrödinger motion outside S and subjected to the boundary condition of an infinite potential barrier in the region S. The solution of the corresponding Schrödinger equation could not be separated in general into an x and a y dependence. However, we approximate the motion of the particle coming from the remote x region by a factorized Gaussian wave packet whose expression is assumed as

$$\psi(x, y, t) = \psi(x, t)\phi(y, t) \tag{3}$$

$$\psi(x, t) = \alpha^{1/2} \frac{\exp\left[-\frac{\alpha^2}{2}\frac{(x - x_0 - \hbar k_{0x} t/m)^2}{1 + i\hbar\alpha^2 t/m} + ik_{0x}(x - x_0) - i\frac{\hbar k_{0x}^2 t}{2m}\right]}{[\pi^{1/2}(1 + i\hbar\alpha^2 t/m)]^{1/2}} \tag{4}$$

$$\phi(y, t) = \left[\frac{\beta}{\pi^{1/2}(1 + i\hbar\beta^2 t/m)}\right]^{1/2} \exp\left[-\frac{\beta^2}{2}\frac{(y - y_0)^2}{1 + i\hbar\beta^2 t/m}\right] \tag{5}$$

To describe the motion of the wave packet after the slits, if no other interactions are considered, we generalize what was done for the cases $N = 1, 2$ (Zecca and Cavalleri, 1997; Zecca, 1999). By the truncation assumption, the wave function $\psi(x, y, t)$ immediatly after the slits, at a time taken as the initial time $t = 0$, is assumed to have the form $\psi_I(x, y, 0) = \psi_a(x, 0)\, \chi_I(y)\phi(y, 0)$, where $\psi_a(x, 0)$ is the function $\psi(x, 0)$ in equation (4) with $x_0 = a$, and $\chi_I(y)$ is the characteristic function of the set I. After the slits a free Schrödinger time evolution with initial state $\psi_I(x, y, 0)$ is assumed. At time t, the wave function has therefore the form

$$\psi_I(x, y, t) = \psi_a(x, t)\phi_I(y, t) \tag{6}$$

where $\psi_a(x, t)$ is again the function in equation (4) with $x_0 = a$, and

$$\phi_I(y, t) = \frac{1}{2\hbar}\frac{\beta^{1/2}}{\pi^{5/4}}\int_{\mathcal{R}} dp_y \exp\left[\frac{i}{\hbar}\left(p_y y - \frac{p_y^2 t}{2m}\right)\right] \times \int_I d\xi \exp\left[-\frac{i}{\hbar}p_y\xi - \frac{\beta^2}{2}(\xi - y_0)^2\right] \tag{7}$$

$$= \left[\frac{m\beta}{\pi^{3/2}2i\hbar t}\right]^{1/2}\exp\left[y^2\frac{im}{2\hbar t} - y_0^2\frac{\beta^2}{2}\right] \times \int_I \exp\left[-\xi^2\left(\frac{\beta^2}{2} - \frac{im}{2\hbar t}\right) + \xi\left(y_0\beta^2 - \frac{imy}{\hbar t}\right)\right]d\xi \tag{8}$$

By integrating over ξ, one gets

$$\phi_I(y, t) = \frac{1}{2}\left[\frac{m\beta}{\pi^{1/2}(m + it\hbar\beta^2)}\right]^{1/2}\exp\left[-\frac{m\beta^2}{2(m + it\hbar\beta^2)}(y - y_0)^2\right] \times \sum_{j=1}^{N}\left\{\operatorname{erf}\left[\frac{im(y - d_j - b_j) - \beta^2\hbar t(y_0 - d_j - b_j)}{(2\hbar t(\hbar t\beta^2 - im))^{1/2}}\right] - \operatorname{erf}\left[\frac{im(y - d_j) - \beta^2\hbar t(y_0 - d_j)}{(2\hbar t(\hbar t\beta^2 - im))^{1/2}}\right]\right\} \tag{9}$$

where $\operatorname{erf} z = (2/\sqrt{\pi})\int_0^z \exp(-t^2)\,dt$ is the error function (Abramovitz and Stegun, 1960).

3. *N*-SLIT QM DIFFRACTION: LIMITING CASES

3.1. Suppose the incoming wave packet is narrow with respect to the slits:

$$\Delta y = \frac{1}{\beta\sqrt{2}} \ll b_j, \qquad j = 0, 1, 2, \ldots, N \tag{10}$$

By considering the dominant term for β large in the argument of the erf function, one gets from (9)

$$\phi_I\phi_I^\star \cong \frac{\pi^{-3/2}\beta m}{(m^2 + \hbar^2 t^2\beta^4)^{1/2}} \times \exp\left[-\frac{m^2\beta^2(y - y_0)^2}{m^2 + \hbar^2 t^2\beta^4}\right]\left|\sum_{j=1}^{N}\int_{(y_0 - d_j - b_j)\beta/\sqrt{2}}^{(y_0 - d_j)\beta/\sqrt{2}} dt\, e^{-t^2}\right|^2 \tag{11}$$

Since the contribution of the integrals in (11) is negligible unless $y_0 \in I$, the situation describes an incoming narrow wave packet that passes essentially undisturbed through the slits or is reflected toward the negative x axis according to the incoming y probability distribution is centered in correspondence to one of the slits or not.

3.2. In the case of an incoming wave packet that is very indeterminate in the y position

$$\Delta y = \frac{1}{\beta\sqrt{2}} \gg b_j, \qquad j = 1, 2, \ldots, N \tag{12}$$

by setting $\beta^2 = 0$ and neglecting the term $-im/(2\hbar t)$, we see that the integral over ξ in (8) takes the value

$$\frac{2\hbar t}{my}\sum_{j=1}^{N} \exp\left[-i\frac{my}{\hbar t}\left(d_j + \frac{b_j}{2}\right)\right]\sin\left(\frac{mb_j}{2\hbar t}y\right) \tag{13}$$

By using this result in (8) one gets

$$\phi_I(y, t)\phi_I^*(y, t) \cong \frac{2\beta\hbar t}{m\pi^{3/2}}\left|\sum_{j=1}^{N} \times \exp\left[-i\frac{my}{\hbar t}\left(d_j + \frac{b_j}{2}\right)\right]\frac{\sin(mb_jy/2\hbar t)}{y}\right|^2 \tag{14}$$

which represents the interference pattern produced by the N slits. It has a quite complicated structure unless the geometry of the slits has some regularity property.

3.3. Suppose now $\beta^2 = 0$, the slits have the same aperture b, and the barriers all have the same dimension $d_{j+1} - (d_j + b_j) = d$, $j = 1, 2, \ldots, N$, so that

$$\begin{aligned} b_j &= b, \qquad j = 1, 2, \ldots, N \\ d_j &= d_1 + (j-1)(d+b) \end{aligned} \tag{15}$$

By using the relations (15), we can calculate the expression (13) to obtain

$$\frac{2\hbar t}{my}\sin\left(\frac{mb}{2\hbar t}y\right)\exp\left[-\frac{im}{\hbar t}\left(d_1 + \frac{b}{2}\right)y\right] \times \frac{1 - \exp[-im(b+d)Ny/\hbar t]}{1 - \exp[-im(b+d)y/\hbar t]} \tag{16}$$

so that from (8), (16) one gets

$$\phi_I \phi_I^\star \cong \frac{b^2 m\beta}{2\pi^{3/2}\hbar t} \exp\,(-y_0^2\beta^2) \left(\frac{\sin\, mby/2\hbar t}{myb/(2\hbar t)} \right)^2 \times \frac{1 - \cos\,[mN(b + d)y/t\hbar]}{1 - \cos\,[my(b + d)/t\hbar]} \tag{17}$$

which represents the interference pattern of N equal slits of apeture b equally separated by a distance $b + d$. The expression (17) reduces for $N = 1, 2$ to the corresponding results previously obtained (Zecca and Cavalleri, 1997; Zecca, 1999).

4. REGULAR LATTICE OF SLITS

The case of equal slits of aperture b equally spaced at a distance $b + d$ by barriers of width d (the geometry of Section 3.3) can be further studied in the case of N very large, practically infinite. Since the incoming wave packet has a nontrivial probability distribution of the y momentum, it seems of interest to consider the effect of the boundary condition given by the periodic structure of the barrier. The effect can be described by the action of an infinite periodic potential in the y direction, which according to the general scheme (Zecca and Cavalleri, 1997), can be seen as the limit $V_0 \to \infty$ of the potential $V(y)$ defined by

$$V(y) = \begin{cases} 0 & \text{if} \quad 0 < y < b \\ V_0 & \text{if} \quad b < y < b + d \end{cases} \tag{18}$$

and of period $b + d$. No other interactions are considered. This implies that near the slits the energy spectrum in the y direction cannot be that of a free particle as assumed in (7), but must be compatible with the band structure of the energy spectrum W of a one-dimensional Schrödinger particle with periodic potential. It is well known that such an energy band structure is determined by the constraint (see e.g., Merzbacher, 1970)

$$\left|\cosh \bar{k}_y d \cos k_y b + \frac{\bar{k}_y^2 - k_y^2}{2\bar{k}_y k_y} \sinh \bar{k}_y\, d \sin k_y b\right| \leq 1 \tag{19}$$

where $\hbar k_y = \sqrt{2mW}$, $\hbar \bar{k}_y = \sqrt{2m|W - V_0|}$ $(W < V_0)$. An elementary study for large V_0 shows that the constraint (19) is satisfied in the neighborhoods of $k_y b = n\pi$, $n = \pm 1, \pm 2, \pm \ldots$, and that in the limit $V_0 \to \infty$ the allowed energy values are

$$W_n = \frac{(\hbar \pi n/b)^2}{2m}, \qquad n = 0, \pm 1, \pm 2, \pm \ldots \tag{20}$$

In the limit of infinite barrier we assume therefore that near the slits the energy spectrum of the particle in the y direction is given by the step function

$$W(p_y) = \frac{(\hbar\pi/b)^2}{2m} \sum_{n=0}^{\infty} n^2 \{\chi_{[n\pi\hbar/b,(n+1)\pi\hbar/b)}(p_y) + \chi_{(-(n+1)\pi\hbar/b,-n\pi\hbar/b]}(p_y)\} \tag{21}$$

$\chi_{[\alpha,\beta)}(p_y)$ is the characteristic function of the interval $[\alpha, \beta)$. To determine the diffraction pattern after the slits, we assume the time evolution scheme of Section 2, where besides the truncation assumption, we assume the energy band spectrum (21) on the initial condition. This amounts in calculating, instead of (7), the expression

$$\phi_I(y, t) = \frac{1}{2\hbar} \frac{\beta^{1/2}}{\pi^{5/4}} \int_{\mathcal{R}} dp_y \exp\left[\frac{i}{\hbar}(p_y y - W(p_y)t)\right] \int_I d\xi \times \exp\left[-\frac{i}{\hbar} p_y \xi - \frac{\beta^2}{2}(\xi - y_0)^2\right] \tag{22}$$

with $W(p_y)$ now given by (21). One gets

$$\phi_I(y, t) = \frac{\beta^{1/2}}{2\hbar\pi^{5/4}} \int_I d\xi \exp\left[-\frac{\beta^2}{2}(\xi - y_0)^2\right] \sum_{n=0}^{\infty} \exp\left[-\frac{it}{2m\hbar}\left(\frac{\pi\hbar}{b}\right)^2 n^2\right] \times \left(\int_{n\pi\hbar/b}^{(n+1)\pi\hbar/b} + \int_{-(n+1)\pi\hbar/b}^{-n\pi\hbar/b}\right) dp_y \exp\left[\frac{i}{\hbar} p_y (y - \xi)\right] \tag{23}$$

and hence by performing the integrations one has the general result

$$\phi_I = \frac{2\beta^{1/2}}{\pi^{5/4}} \int_I d\xi \exp\left[-\frac{\beta^2}{2}(\xi - y_0)^2\right] \sum_{n=0}^{\infty} \left\{ \exp\left[-\frac{it}{2m\hbar}\left(\frac{\pi\hbar}{b}\right)^2 n^2\right] \times \frac{\sin(\pi(y - \xi)/2b)}{y - \xi} \cos\left[\left(n + \frac{1}{2}\right)\frac{\pi}{b}(y - \xi)\right]\right\} \tag{24}$$

The last integral can be calculated in the usual limiting situations.

4.1. Suppose $\Delta y = 1/\beta\sqrt{2} \ll b$ or $\beta \to \infty$. Since $\exp[-\beta^2(\xi - y_0)^2/2]$ is proportional, for large β, to $\delta(\xi - y_0)$, then if $y_0 \notin I$, one gets zero from (24), while if $y_0 \in I$, one obtains

$$\phi_I\phi_I^\star \cong \frac{2\pi^{1/2}}{\beta b^2}\frac{\sin^2(\pi/2b)(y - y_0)}{[(\pi/2b)(y - y_0)]^2} \times \left|\sum_{n=0}^{\infty}\exp\left[-\frac{it\hbar\pi^2}{2mb^2}n^2\right]\cos\left[\left(n + \frac{1}{2}\right)\frac{\pi}{b}(y - y_0)\right]\right|^2 \tag{25}$$

If $y_0 \in I$, the diffraction pattern therefore has at time t the profile (25) in the y direction, which has a complex structure, but that for b small is still peaked around $y = y_0$.

4.2. Suppose now $\Delta y = 1/\beta\sqrt{2} \gg b$ or β small. By setting $\beta^2 = 0$, the expression (24) becomes

$$\phi_I = \frac{\beta^{1/2}}{\pi^{5/4}}\sum_{n=0}^{\infty}\exp\left[-\frac{it\hbar\pi^2}{2mb^2}n^2\right] \times \sum_j\int_{d_j}^{d_j+b}d\xi\,\frac{\sin[(n+1)(\pi/b)(y-\xi)] + \sin[n(\pi/b)(y-\xi)]}{y-\xi} \tag{26}$$

or, in terms of the sine integral $\mathrm{Si}(z) = \int_0^z dz \sin z/z$,

$$\phi_I = \frac{\beta^{1/2}}{\pi^{5/4}}\sum_{n=0}^{\infty}\exp\left[-\frac{it\hbar\pi^2}{2mb^2}n^2\right]\sum_j\left\{\mathrm{Si}\left[(d_j + b - y)(n+1)\frac{\pi}{b}\right] - \mathrm{Si}\left[(d_j - y)(n+1)\frac{\pi}{b}\right] + \mathrm{Si}\left[(d_j + b - y)\,n\frac{\pi}{b}\right] - \mathrm{Si}\left[(d_j - y)n\frac{\pi}{b}\right]\right\} \tag{27}$$

By considering the qualitative behavior of the sine integral function (Abramovitz and Stegun, 1960), it appears that the dominant contribution in equation (27) comes from the lowest n values. The contribution relative to $n = 0$ can be obtained directly from (24) with $\beta^2 = 0$:

$$\phi_I = \frac{\beta^{1/2}}{\pi^{5/4}}\sum_j\int_{d_j}^{d_j+b}d\xi\,\frac{\sin(\pi/b)(y-\xi)}{y-\xi} \tag{28}$$

and for b small

$$\phi_I = \frac{\beta^{1/2}}{\pi^{5/4}} b \sum_j \frac{\sin(\pi/b)(y - d_j - b/2)}{y - d_j - b/2} \tag{29}$$

which represents an interference of elementary amplitudes of single slit.

5. REMARKS

In the previous sections the diffraction pattern of particles through N slits has been calculated based on the pure truncation assumption and Schrödinger wave propagation. In case of N large the calculations have been performed by also taking into account the periodic boundary structure of the barrier. This has been done by giving to the particle an energy band structure in the y direction when passing the slits. The results obtained are given in (9) and (24), respectively, and are difficult to compare in general. In the case of an incoming wave packet with Δy large with respect to the slit aperture the results obtained in the two different ways are given in (17) and (27). The essential feature is that in equation (17) (absence of periodic potential) there is a dominating structure typical of the diffraction pattern through a single slit, while in the case of a periodic potential this structure is present corresponding to every slit [compare also with equations (28), (29)].

In the case of a beam of particles with $\Delta y \ll b$ the different patterns are given equations (11), (25). As expected, in both cases the diffraction pattern is peaked around the correct position, but in equation (11) the profile is Gaussian-like, while in equation (25) it is similar to that through a single slit. The results make questionable the correctness of the energy band structure approach, which, however, has the strength of being based on boundary conditions. It would be of interest to decide whether the periodic potential assumption has an experimental counterpart.

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